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ON THE STRESS ANALYSIS OF AN ELLIPTIC PLATE BY THE METHOD OF FILONENKO-BORODICH

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Abstract—The stress distribution due to an arbitrary external in-plane load on the edges of a thin elastic plate has been studied. An approximate solution to the problem has been obtained by a method due to Filonenko-Borodich who used it originally for solving the corresponding problems for a rectangular plate and for the problem of the equilibrium of an elastic parallelepiped.

1. INTRODUCTION

THIS paper deals with an approximate solution for the stress distribution at any point of a thin elastic elliptic plate whose edges are subjected to an arbitrary load by a method due to Filonenko-Borodich [1] which was used by Singh [2] for solving a three-dimensional problem of an elliptic prism.

In Section 1, a brief resume of the general method is included. Section 2 deals with the application of the method to the case of an elliptic plate. As an illustration of the procedure, the method has been applied in Section 3 to a particular numerical example. Since the method represents the solution in the form of an infinite series, which involves a good deal of operations of a simple nature, punch-card machines were employed in obtaining the numerical results given in this paper. All details of the example are explained by graphs.

2. THE METHOD

The problem to be treated in this paper is that of a plate of elliptical boundary. Hence we consider throughout the elliptic co-ordinates (ξ, η) which are related to the cartesian co-ordinates (x, y) by

$$x+iy = c\cosh(\xi+i\eta), \qquad c > 0. \tag{2.1}$$

Thus

$$x = c\cosh\xi\cos\eta, \qquad y = c\sinh\xi\sin\eta \tag{2.2}$$

and

$$\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \xi}\right)^2 = \left(\frac{\partial x}{\partial \eta}\right)^2+\left(\frac{\partial y}{\partial \eta}\right)^2 = c^2k \tag{2.3}$$

where

$$k = \sinh^2\xi+\sin^2\eta. \tag{2.4}$$

The equations $\xi =$ const. and $\eta =$ const. represent systems of confocal ellipses and hyperbolas respectively. The centre of any ellipse $\xi =$ const. has the elliptic co-ordinates $(0, \pi/2)$ and the focii have the co-ordinates $(0, 0)$ and $(0, \pi)$. The cartesian equation of the bounding ellipse is taken as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \qquad a > b > 0. \tag{2.5}$$

The boundary will be expressed in elliptic co-ordinates by $\xi = \alpha > 0$ where α is determined by

$$a = c\cosh\alpha, b = c\sinh\alpha \quad \text{and} \quad c^2 = a^2-b^2. \tag{2.6}$$

It is well known that in plane problems the stress components τ_{ij} in cartesian co-ordinates are derived from a single stress function ψ such that

$$\tau_{xx} = \frac{\partial^2\psi}{\partial y^2}; \qquad \tau_{yy} = \frac{\partial^2\psi}{\partial x^2}; \qquad \tau_{xy} = -\frac{\partial^2\psi}{\partial x\partial y}$$

where ψ satisfies the biharmonic equation

$$\nabla^4\psi = 0.$$

In elliptic co-ordinates the corresponding stress components are given by [3].

$$\begin{aligned}
c^2\tau_{\xi\xi} &= \frac{1}{k}\frac{\partial^2\psi}{\partial\eta^2}+\frac{1}{2k^2}\left(\frac{\partial\psi}{\partial\xi}\sinh 2\xi-\frac{\partial\psi}{\partial\eta}\sin 2\eta\right)\\
c^2\tau_{\xi\eta} &= \frac{1}{k}\frac{\partial^2\psi}{\partial\xi\partial\eta}+\frac{1}{2k^2}\left(\frac{\partial\psi}{\partial\xi}\sin 2\eta+\frac{\partial\psi}{\partial\eta}\sinh 2\xi\right)\\
c^2\tau_{\eta\eta} &= \frac{1}{k}\frac{\partial^2\psi}{\partial\xi^2}+\frac{1}{2k^2}\left(\frac{\partial\psi}{\partial\eta}\sin 2\eta-\frac{\partial\psi}{\partial\xi}\sinh 2\xi\right)
\end{aligned} \tag{2.7}$$

where ψ satisfies the biharmonic equation in elliptic co-ordinates, namely,

$$\nabla^4\psi = \frac{1}{c^4}\left[\frac{1}{k^2}\nabla_1^4-\frac{2}{k^3}\left(\sinh 2\xi\frac{\partial}{\partial\xi}+\sin 2\eta\frac{\partial}{\partial\eta}\right)\nabla_1^2+\frac{\sinh^2 2\xi+\sin^2 2\eta}{k^4}\nabla_1^2\right]\psi = 0 \tag{2.8}$$

and that

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}, \qquad \nabla_1^4 \equiv \nabla_1^2\nabla_1^2.$$

We shall suppose that on the boundary of the plate $\xi = \alpha$ act forces of the general type given by

$$\tau_{\xi\xi}|_{\xi=\alpha} = f(\eta), \qquad \tau_{\xi\eta}|_{\xi=\alpha} = g(\eta) \tag{2.9}$$

where each of the functions $f(\eta)$ and $g(\eta)$ can be expressed as Fourier series

$$\begin{aligned}
f(\eta) &= \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\eta+b_n\sin n\eta),\\
g(\eta) &= \frac{d_0}{2}+\sum_{n=1}^{\infty}(d_n\cos n\eta+c_n\sin n\eta).
\end{aligned} \tag{2.10}$$

If we can find a stress function $\varphi(\xi, \eta)$ which satisfies the equation (2.8) and that the stress-components derived from it according to (2.7) satisfy the boundary conditions (2.9) then we can determine the stress distribution at any point of the plate completely.

However, in practice, it is difficult to find a function $\varphi(\xi, \eta)$ satisfying both (2.8) and (2.9). Therefore, following Filonenko-Borodich we will express the function φ as the sum of two functions in the form

$$\varphi = \varphi_b(\xi, \eta) + \varphi_c(\xi, \eta). \tag{2.11}$$

The function φ_b is called the basic function and the function φ_c is called the correcting function. We require that the functions φ_b and φ_c are such so that the stress components $\tau_{\xi\xi}^{(b)}$ and $\tau_{\xi\eta}^{(b)}$ derived from φ_b reduce to the given values $f(\eta)$ and $g(\eta)$ respectively on the boundary $\xi = \alpha$ where as the stress components $\tau_{\xi\xi}^{(c)}$ and $\tau_{\xi\eta}^{(c)}$ derived from the function φ_c vanish on the boundary. Consequently, the functions $\varphi_b(\xi, \eta)$ and $\varphi_c(\xi, \eta)$ being summed satisfy the boundary conditions (2.9) of the problem.

The construction of the basic function φ_b offers no difficulty in principle but it is found to be more or less involved depending on the complexity of prescribed boundary loads. However, the construction of φ_c does not depend on the given load since stress components derived from it must vanish on the boundary; hence it can be constructed once and for all for a given plate. In general, the functions φ_b and φ_c do not satisfy the biharmonic equation (2.8). If they do so, then the exact solution of the problem is obtained. On the other hand, we stipulate to obtain an approximate solution by choosing φ_c in such a way so that it contains linearly several arbitrary constants admitting the variation of the function φ_c and permitting the state of stress of the plate to be varied so as to satisfy the boundary conditions. The greater the number of terms we introduce in φ_c the closer we come to the satisfaction of the equation (2.8) and hence the more closely we come to the exact solution. The constant co-efficients in φ_c can be determined by any of the direct methods in the variational calculus.

3. DISCUSSION

We now discuss, in detail, the method described in the previous section and outline a technique for the construction of the basic and correcting functions. The correcting function $\varphi_c(\xi, \eta)$ is to be constructed such that the stress components $\tau_{\xi\xi}^{(c)}$ and $\tau_{\xi\eta}^{(c)}$ derived from it must vanish on the boundary $\xi = \alpha$ of the plate. This condition is, however, fulfilled if $\varphi_c(\xi, \eta)$ satisfies

$$\varphi_c(\xi, \eta)\bigg|_{\xi=\alpha} = \frac{\partial}{\partial \xi}\varphi_c(\xi, \eta)\bigg|_{\xi=\alpha} = \frac{\partial^2}{\partial \xi^2}\varphi_c(\xi, \eta)\bigg|_{\xi=\alpha} = 0 \tag{3.1}$$

which is apparent from (2.7). The function $\varphi_c(\xi, \eta)$ satisfying all the required conditions, may be conveniently constructed in the form

$$\varphi_c = \sum_m \sum_n A_{mn} k^3 \sin \frac{\pi}{2\alpha}(\xi + \alpha) P_m(\xi) P_n(\eta) \tag{3.2}$$

where $P_m(\xi)$ and $P_n(\eta)$ are so called cosine-binomials [1]

$$\begin{aligned} P_m(\xi) &= \cos\frac{m\pi}{2\alpha}(\xi+\alpha)-\cos(m+2)\frac{\pi}{2\alpha}(\xi+\alpha), \qquad 0 \le \xi \le \alpha \\ P_n(\eta) &= \cos n\eta-\cos(n+2)\eta, \qquad 0 \le \eta \le 2\pi \\ m, n &= 0, 2, 4, \ldots \end{aligned} \tag{3.3}$$

and A_{mn} are the arbitrary constant coefficients which can be determined by using any of the direct methods.

We shall, however, use here the Galerkin method for the determination of these parameters. The cosine-binomials satisfy the required boundary conditions. Systems of these functions are complete and closed. If we derive the stress components in (2.7) by means of the correcting function (3.2) then we note that as a consequence of the boundary properties of the cosine-binomials the entire contour of the plate will be free of stresses. The completeness of the function system (3.3) and the arbitrariness of the coefficients A_{mn} allow one to realize an extremely wide class of functions by this means.

Returning now to the construction of the basic function φ_b it is apparent that its construction is much more difficult in the elliptic co-ordinate system than in the cartesian system. Since the basic stress function is to be constructed in such a manner so that the stress components $\tau_{\xi\xi}^{(b)}$ and $\tau_{\xi\eta}^{(b)}$ derived from it according to (2.7) must satisfy the prescribed boundary conditions on the contour $\xi = \alpha$ of the plate, one of the possible ways of its construction is to seek it in the form of the series

$$\varphi_b = \sum_n C_n\varphi_n \tag{3.4}$$

where C_n are constants to be determined from the prescribed boundary conditions and φ_n are suitably chosen functions which may be harmonic, biharmonic or the combination of both or even arbitrary. We will come to this discussion in more detail in the next section. Substituting the expression (3.2) and (3.4) in (2.8) we obtain

$$\nabla^4\varphi = \nabla^4(\varphi_b+\varphi_c) = \varepsilon(\xi, \eta\,; A_{mn}) \tag{3.5}$$

where $\varepsilon(\xi, \eta\,; A_{mn})$ which is, in general, not identically equal to zero can be viewed as an error function. The arbitrary constant coefficients A_{mn} appearing in the error function can be determined by using Galerkin's orthogonality conditions

$$\int\!\!\int_\Omega \varepsilon(\xi, \eta\,; A_{mn})k^3 \sin\frac{\pi}{2\alpha}(\xi+\alpha)P_i(\xi)P_j(\eta)\,\mathrm{d}\Omega = 0 \qquad i, j = 0, 2, 4, \ldots \tag{3.6}$$

which in this can be written as

$$\int_0^\alpha\int_0^{2\pi} [L\{\varphi_c(\xi, \eta)\}]k^4 \sin\frac{\pi}{2\alpha}(\xi+\alpha)P_i(\xi)P_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = -\int_0^\alpha\int_0^{2\pi} [L\{\varphi_b(\xi, \eta)\}]k^4 \sin\frac{\pi}{2\alpha}(\xi+\alpha)P_i(\xi)P_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.7}$$

where we have used the notation

$$L \equiv \frac{1}{c^4}\left[\frac{1}{k^2}\nabla_1^4-\frac{2}{k^3}\left(\sinh 2\xi\frac{\partial}{\partial\xi}+\sin 2\eta\frac{\partial}{\partial\eta}\right)\nabla_1^2+\frac{\sinh^2 2\xi+\sin^2 2\eta}{k^4}\right]. \tag{3.8}$$

4. ILLUSTRATION

As an illustration of the procedure, let us consider the following particular problem. Suppose that the edge of the plate $\xi = \alpha$ is acted upon by compressive forces of the type (Fig. 1)

$$\begin{aligned} f(\eta) &= -(\lambda+\mu\cos 2\eta), \qquad \lambda > \mu \geq 0 \\ g(\eta) &= 0. \end{aligned} \tag{4.1}$$

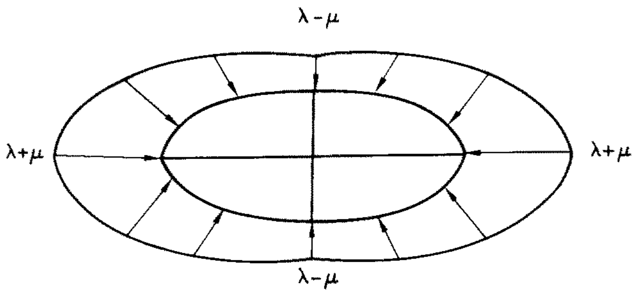

FIG. 1.

In this case, we have

$$\begin{aligned} \tau_{\xi\xi}|_{\xi=\alpha} &= -(\lambda+\mu\cos 2\eta) \\ \tau_{\xi\eta}|_{\xi=\alpha} &= 0. \end{aligned} \tag{4.2}$$

In order to determine the basic stress function φ_b corresponding to the boundary conditions (4.2), we take φ_b in the form

$$\varphi_b = \sum_{n=0}^{6} C_n \varphi_n \tag{4.3}$$

where $\varphi_0, \ldots, \varphi_6$ are chosen as follows

$$\begin{aligned} \varphi_0 &= (3\cosh 2\alpha\cosh 4\alpha - 4\cosh 4\alpha\cosh 2\xi + \cosh 2\alpha\cosh 4\alpha) \\ &\quad (3\cosh 2\alpha\cosh 4\alpha - 4\cosh 4\alpha\cos 2\eta + \cosh 2\alpha\cos 4\eta) \\ \varphi_1 &= (\cosh 2\xi - \cosh 2\alpha)(\cosh 2\alpha - \cos 2\eta) \\ \varphi_2 &= (\cosh 4\alpha\cosh 2\xi - \cosh 2\alpha\cosh 4\xi)(\cosh 2\alpha - \cos 2\eta) \\ &\quad (1+2\cosh 2\alpha\cos 2\eta) \\ \varphi_3 &= (\cosh 2\alpha - \cosh 2\xi)(\cosh 2\alpha - \cos 2\eta)(\cosh 2\xi + \cos 2\eta) \\ \varphi_4 &= \cosh 2\xi + \cos 2\eta \\ \varphi_5 &= \cosh 4\xi + \cos 4\eta \\ \varphi_6 &= \cosh 4\xi\cos 4\eta \end{aligned} \tag{4.4}$$

and C_n are unknown constants to be determined from the boundary conditions. It is to be mentioned that most of the functions in (4.4) have previously been used by Galerkin [4] for the analysis of deflection of an elliptic plate.

The boundary conditions (4.2) reduce to

$$-\frac{c^2}{2}(\lambda+\mu\cos 2\eta)(\cosh 2\alpha-\cos 2\eta)^2$$

$$=(\cosh 2\alpha-\cos 2\eta)\left\{\frac{\partial^2\varphi_b}{\partial\eta^2}\right\}_\alpha+\sinh 2\alpha\left\{\frac{\partial\varphi_b}{\partial\xi}\right\}_\alpha-\sin 2\eta\left\{\frac{\partial\varphi_b}{\partial\eta}\right\}_\alpha \tag{4.5}$$

$$-(\cosh 2\alpha-\cos 2\eta)\left\{\frac{\partial^2\varphi_b}{\partial\xi\partial\eta}\right\}_\alpha+\sin 2\eta\left\{\frac{\partial\varphi_b}{\partial\xi}\right\}_\alpha+\sinh 2\alpha\left\{\frac{\partial\varphi_b}{\partial\eta}\right\}_\alpha=0. \tag{4.6}$$

After substituting (4.3) in (4.5) and (4.6) using the expressions (4.4), we obtain two equations in sines and cosines of η. Since these two equations must be true for all values of η, equating the constant member, the coefficients of various cosine and sine terms we obtain only seven equations [four from (4.5) and three from (4.6)] for the determination of seven unknown constants $c_0, c_1, \ldots, c_6$. For example, the equation which corresponds to equating the constant member in (4.5) is given by

$$\tfrac{3}{4}C_0(-\cosh 14\alpha+3\cosh 10\alpha-3\cosh 6\alpha+\cosh 2\alpha)+\tfrac{1}{2}C_1(\cosh 6\alpha-\cosh 2\alpha)$$
$$-\tfrac{1}{4}C_3(\cosh 8\alpha-2\cosh 4\alpha+1)+C_4(2+\cosh 4\alpha)+2C_5(\cosh 6\alpha-\cosh 2\alpha)$$
$$=\frac{c^2}{4}\{2\mu\cosh 2\alpha-\lambda(2+\cosh 4\alpha)\}. \tag{4.7}$$

Similarly, equating the coefficient of $\sin 2\eta$ in (4.6) equal to zero, we obtain

$$2C_0(2\sinh 4\alpha-\sinh 8\alpha)+C_2(\sinh 8\alpha+4\sinh 4\alpha)+2C_3\sinh 2\alpha+8C_6\sinh 4\alpha=0. \tag{4.8}$$

Solving the seven equations obtained in this way, we finally get

$$C_0=0, \tag{4.9}$$

$$C_1=\frac{1}{J}\cdot\frac{M}{\sinh 2\alpha}(-120\sinh 14\alpha+228\sinh 12\alpha-264\sinh 10\alpha+264\sinh 8\alpha$$
$$-108\sinh 6\alpha+12\sinh 4\alpha+36\sinh 2\alpha)-\frac{c^2}{12}\mu, \tag{4.10}$$

$$C_2=-\frac{48}{J}M(\cosh 8\alpha-\cosh 6\alpha+\cosh 4\alpha-\cosh 2\alpha), \tag{4.11}$$

$$C_3=\frac{4}{J}\cdot\frac{M}{\sinh 2\alpha}(\tfrac{7}{2}\sinh 16\alpha-3\sinh 14\alpha+17\sinh 12\alpha-15\sinh 10\alpha+11\sinh 8\alpha$$
$$-15\sinh 6\alpha-15\sinh 4\alpha-3\sinh 2\alpha), \tag{4.12}$$

$$C_4=\frac{M}{J}(3\cosh 18\alpha-3\cosh 16\alpha+12\cosh 14\alpha-12\cosh 12\alpha-3\cosh 10\alpha$$
$$-21\cosh 6\alpha+12\cosh 4\alpha+9\cosh 2\alpha+9)+\frac{c^2}{12}(2\mu\cosh 2\alpha-3\lambda), \tag{4.13}$$

$$C_5 = \frac{M}{J}(\cosh 12\alpha + 4\cosh 8\alpha - \cosh 4\alpha - 4)\cosh 4\alpha - \frac{c^2}{48}\mu, \tag{4.14}$$

$$C_6 = \frac{M}{J}(-\cosh 12\alpha - 4\cosh 8\alpha + \cosh 4\alpha + 4), \tag{4.15}$$

in which the notations

$$\begin{aligned} J = {} & 24(-\cosh 10\alpha + \cosh 8\alpha - 4\cosh 6\alpha + 4\cosh 4\alpha - \cosh 2\alpha - 1) \\ & .(-\cosh 12\alpha - 4\cosh 8\alpha + \cosh 4\alpha + 4) \\ & -48(\cosh 14\alpha + \cosh 10\alpha - 9\cosh 6\alpha + 7\cosh 2\alpha) \\ & .(\cosh 8\alpha - \cosh 6\alpha + \cosh 4\alpha - \cosh 2\alpha), \end{aligned} \tag{4.16}$$

$$M = \tfrac{2}{3}.\mu c^2(\cosh 2\alpha - \cosh 6\alpha)$$

were employed.

5. A NUMERICAL EXAMPLE

Let us consider a numerical example illustrating the theoretical solutions of the previous section by taking $\alpha = 1$.

The basic stress function φ_b is written as

$$\varphi_b = \sum_{n=1}^{6} \bar{C}_n \bar{\varphi}_n \tag{5.1}$$

where $\bar{C}_n$ and $\bar{\varphi}_n$ are the expressions for C_n and φ_n respectively obtained after replacing α by unity. The correcting stress function φ_c becomes

$$\varphi_c = \sum_m \sum_n A_{mn} k^3 \sin\frac{\pi}{2}(1+\xi)\bar{P}_m(\xi)P_n(\eta) \tag{5.2}$$

where

$$\bar{P}_m(\xi) = \cos\frac{m\pi}{2}(1+\xi) - \cos(m+2)\frac{\pi}{2}(1+\xi). \tag{5.3}$$

Substituting φ_b and φ_c given by (5.1) and (5.2) in (3.7) and noting that

$$L(\bar{\varphi}_1 + \bar{\varphi}_2 + \bar{\varphi}_4 + \bar{\varphi}_6) = 0 \tag{5.4}$$

and

$$L(\bar{C}_3\bar{\varphi}_3 + \bar{C}_5\bar{\varphi}_5) = \frac{256}{c^4}(2\bar{C}_5 - \bar{C}_3 \cosh 2) \tag{5.5}$$

the equation (3.7) transforms into

$$\begin{aligned} & -\frac{256}{c^4}(2\bar{C}_5 - \bar{C}_3\cosh 2)\int_0^1\int_0^{2\pi} k^4 \sin\frac{\pi}{2}(1+\xi)\bar{P}_i(\xi)P_j(\eta)\,d\xi\,d\eta \\ & = \sum_m\sum_n A_{mn}\int_0^1\int_0^{2\pi}[L\{k^3\sin\frac{\pi}{2}(1+\xi)\bar{P}_m(\xi)P_n(\eta)\}]k^4\sin\frac{\pi}{2}(1+\xi)\bar{P}_i(\xi)P_j(\eta)\,d\xi\,d\eta \end{aligned} \tag{5.6}$$

$$m, n, i, j = 0, 2, \ldots$$

from which the constants A_{mn} are determined.

Considering the first approximation by taking $m = n = 0$, the equation for the determination of A_{00} is obtained by putting $m = n = i = j = 0$ in (5.6) which after integration and calculation leads to

$$-550{\cdot}284001\frac{A_{00}}{c^4} = 500{\cdot}558028\frac{\mu}{c^2} \tag{5.7}$$

whence

$$A_{00} = 0{\cdot}909635\mu c^2. \tag{5.8}$$

Thus the stress components at any point of the plate to the first approximation are obtained.

For the second incomplete approximation we consider only two constants, namely, A_{00} and A_{22} by taking $m = n = 0, 2$. The two equations to determine A_{00} and A_{22} from (5.6) are

$$\begin{aligned}
&-\frac{256}{c^4}(2\bar{C}_5-\bar{C}_3\cosh 2)\int_0^1\int_0^{2\pi} k^4\sin\frac{\pi}{2}(1+\xi)\bar{P}_0(\xi)P_0(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad=\int_0^1\int_0^{2\pi}\left[A_{00}[L\{k^3\sin\frac{\pi}{2}(1+\xi)\bar{P}_0(\xi)P_0(\eta)\}]+A_{22}[L\{k^3\sin\frac{\pi}{2}(1+\xi)\bar{P}_2(\xi)P_2(\eta)\}]\right]\\
&\quad\quad.\,k^4\sin\frac{\pi}{2}(1+\xi)\bar{P}_0(\xi)P_0(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
&-\frac{256}{c^4}(2\bar{C}_5-\bar{C}_3\cosh 2)\int_0^1\int_0^{2\pi} k^4\sin\frac{\pi}{2}(1+\xi)\bar{P}_2(\xi)P_2(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad=\int_0^1\int_0^{2\pi}\left[A_{00}[L\{k^3\sin\frac{\pi}{2}(1+\xi)\bar{P}_0(\xi)P_0(\eta)\}]+A_{22}[L\{k^3\sin\frac{\pi}{2}(1+\xi)\bar{P}_2(\xi)P_2(\eta)\}]\right]\\
&\quad\quad.\,k^4\sin\frac{\pi}{2}(1+\xi)\bar{P}_2(\xi)P_2(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned} \tag{5.10}$$

from which after integration and calculation we have

$$\frac{1}{c^4}\{-550{\cdot}284001A_{00}-520{\cdot}002679A_{22}\} = 500{\cdot}558028\frac{\mu}{c^2} \tag{5.11}$$

and

$$\frac{1}{c^4}\{151{\cdot}10128A_{00}+660{\cdot}123092A_{22}\} = -101{\cdot}868527\frac{\mu}{c^2}. \tag{5.12}$$

Solving (5.11) and (5.12) for A_{00} and A_{22} we obtain

$$\left.\begin{aligned}A_{00} &= -0{\cdot}974624\mu c^2\\ A_{22} &= 0{\cdot}0687726\mu c^2\end{aligned}\right. \tag{5.13}$$

Thus the stress components to the second incomplete approximation can be written down as in the case of first approximation.

It is interesting to note that when $\mu = 0$, we arrive at the case of uniform in-plane compressive forces $(-\lambda)$ i.e. the plate is under two-dimensional hydrostatic loading in its plane. In this case, from (4.9)–(4.15) and (5.13), we have

$$\begin{aligned} \bar{C}_1 &= \bar{C}_2 = \bar{C}_3 = \bar{C}_5 = \bar{C}_6 = A_{00} = A_{22} = 0, \\ \bar{C}_4 &= \frac{-c^2\lambda}{4} \end{aligned} \tag{5.14}$$

and that the stress components at any point in the plate are given by

$$\tau_{\xi\xi} = -\lambda, \qquad \tau_{\xi\eta} = 0, \qquad \tau_{\eta\eta} = -\lambda \tag{5.15}$$

as expected.

The distribution of stress components on an internal ellipse $\xi = \frac{1}{2}$ are plotted against η in Figs. 2–4 for $\lambda = 2\,\mu$.

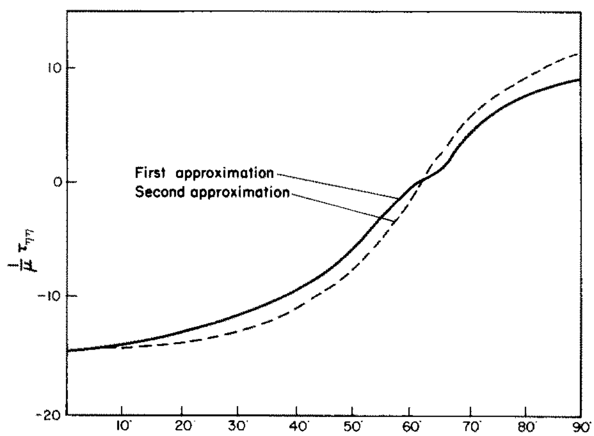

FIG. 2.

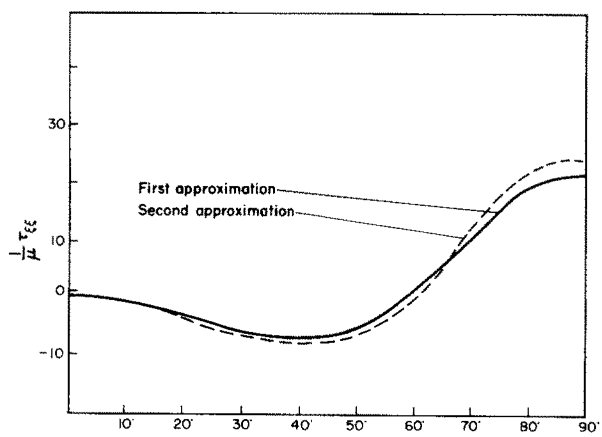

FIG. 3.

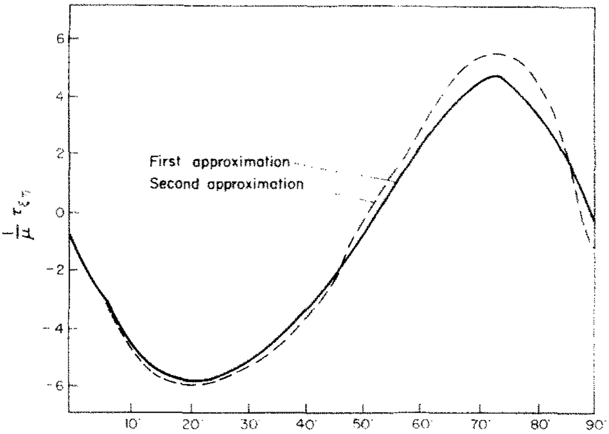

FIG. 4.

6. CONCLUDING REMARKS

From the foregoing analysis it is clear that the method outlined in this paper may be applied to the more general case of an elliptical plate subject to any forces acting on its edges. The analytical procedure presented here is straightforward. In the case of other distributions of tractions over the boundary $\xi = \alpha$ of the plate, we have only to change the form of the basic function φ_b. The form of the correcting function remains the same.

The functions φ_n appearing in the expression (3.4) for φ_b can be constructed arbitrarily but care should be taken such that they do not create any difficulty in the integrations when substituted in (3.7). In fact, the functions we have used for φ_n in (4.4) are due to Galerkin which are very useful and can be used for other types of loadings in conjunction with suitably chosen additional functions according to the requirement.

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Абстракт—Исследуется распределение напряжений, вследствие произвольной внешней нагрузки, действующей в плоскости, на краях тонкой упругой пластинки.

Получается приближенное решение, применяя метод филоненко-бородича, который впервые испоплльзовад этот метод для расчета соотвествующих задач прямоугольной пластинки и для задачи равновесия упругого нараллепипеда.